

DAMPING ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS WITH FINITE TYPE SINGULARITIES

ANDREW COMECH

*Department of Mathematics
University of Toronto
Toronto, Canada M5S 3G3*

ABSTRACT. We derive damping estimates and asymptotics of L^p operator norms for oscillatory integral operators with finite type singularities. The methods are based on incorporating finite type conditions into L^2 almost orthogonality technique of Cotlar-Stein.

1. INTRODUCTION AND RESULTS

The oscillatory integral operators have the form

$$(1.1) \quad T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) u(\vartheta) d\vartheta, \quad x, \vartheta \in \mathbb{R}^n,$$

with $S \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\psi \in C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We denote $h(x, \vartheta) = \det S_{x\vartheta}$. If $h(x, \vartheta) \neq 0$, then it is well-known that the L^2 operator norm of T_λ decays as $\lambda^{-\frac{n}{2}}$ [Hö⁷¹]. The operators with non-empty critical variety

$$\Sigma = \{(x, \vartheta) \mid h(x, \vartheta) = 0\}$$

attracted much attention during last several years: [PaSo⁹⁰], [Pa⁹¹], [PhSt⁹¹] – [PhSt⁹⁷], [GrSe⁹⁴] – [GrSe^{97b}], [Cu⁹⁷]. We recommend [Ph⁹⁵] as a survey on integral operators associated to singular canonical relations.

The properties of T_λ are being characterized in terms of the projections from the associated canonical relation $\mathcal{C} = \{(x, S_x, \vartheta, S_\vartheta)\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ onto the left and right factors. We consider these projections as lifted onto $\mathbb{R}^n \times \mathbb{R}^n$:

$$\pi_L : (x, \vartheta) \mapsto (x, S_x), \quad \pi_R : (x, \vartheta) \mapsto (\vartheta, S_\vartheta).$$

These maps become singular on the critical variety Σ .

We already know [Co⁹⁸] that if one of the projections from the canonical relation is a Whitney fold while the type of the other projection is at most k ($k = 1$ for a Whitney fold), then $\|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + (4 + \frac{2}{k})^{-1}}$. This result was used to obtain the optimal regularity of Fourier integral operators with one-sided Whitney folds.

In this paper, we approach a much more complicated situation when both projections from the canonical relation are of finite type. We develop the machinery which yields the asymptotics of the norm of (1.1) with the integral kernel localized to the region where $h(x, \vartheta) \sim \hbar$, \hbar being some small real number. We then derive the damping estimates on oscillatory integral operators: we will prove that if the symbol of the operator vanishes as $|h(x, \vartheta)|$ on the critical variety, then the operator has such properties as though it is associated to a canonical graph: $\|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2}}$. This result was previously proved for operators associated to two-sided Whitney folds [MeT⁸⁵] and for operators in $n = 1$, with polynomial phases [PhSt⁹⁴]. Damping for operators with one-sided Whitney folds follows from [Co⁹⁷]. A much more general situation (when no assumptions on the projections π_L , π_R are made) is considered in [SoSt⁸⁶]: the damping occurs if the symbol vanishes as $|h(x, \vartheta)|^{5n/2}$.

We will exploit the concept of the type of a map, which we define as the highest order of vanishing of the determinant of its Jacobi matrix in the “critical” direction [Co⁹⁸]. Let M and N be two C^∞ manifolds of the same dimension and let $\pi : M \rightarrow N$ be a smooth map with corank at most 1. Assume that $\det d\pi$ vanishes simply on $\Sigma \subset M$.

Definition. Let \mathbf{V} be any smooth vector field which generates (locally) the kernel of $d\pi$: $\mathbf{V}|_\Sigma \in \text{Ker } d\pi$, $\mathbf{V}|_\Sigma \neq 0$. The type of π at a point $p_0 \in \Sigma$ is defined to be the smallest integer k such that $\mathbf{V}^k \det d\pi|_{p_0} \neq 0$.

The type of π at $p \notin \Sigma$ is defined to be 0.

An example of a map of type at most k is a map which has a Morin S_{1_k} -singularity [Mo⁶⁵]. In particular, the Whitney fold is of type at most 1.

Asymptotics of L^2 estimates. Let us localize the integral kernel of T_λ with the aid of a certain smooth function β to the region where $\det S_{x\vartheta}$ takes the values of size \hbar (for simplicity, we assume that $\hbar > 0$):

$$(1.2) \quad T_\lambda^\hbar u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta)) u(\vartheta) d\vartheta, \quad \beta \in C_{\text{comp}}^\infty([\frac{1}{2}, 2]).$$

We assume that the corank of the mixed Hessian $S_{x\vartheta}$ in (1.1) (and hence the dimension of kernels of $d\pi_L$, $d\pi_R$) is at most 1.

Terminology. If the map π_L is of type at most l , then we will say that the operator T_λ has a singularity of type at most l on the left.

Similarly with the singularity on the right.

Theorem 1.1. *Let T_λ be an oscillatory integral operator of the form (1.2). We assume that $\text{rank } S_{x\vartheta} \geq n-1$ and that T_λ has singularities of type at most l on the left and at most r on the right; we denote $k = \min(l, r)$, $K = \max(l, r)$.*

There are the following estimates on the $L^2 \rightarrow L^2$ action of T_λ^h :

$$(1.3) \quad \|T_\lambda^h\| \leq \text{const } \lambda^{-\frac{n}{2}} h^{-\frac{1}{2}}, \quad h \geq \lambda^{-\frac{1}{3}},$$

$$(1.4) \quad \|T_\lambda^h\| \leq \text{const } \lambda^{-\frac{n+1}{2}} h^{-2}, \quad \lambda^{-(2+\frac{1}{k})^{-1}} \leq h \leq \lambda^{-\frac{1}{3}},$$

$$(1.5) \quad \|T_\lambda^h\| \leq \text{const } \lambda^{-\frac{n}{2}} h^{-1+\frac{1}{2k}}, \quad \lambda^{-(2+\frac{1}{K})^{-1}} \leq h \leq \lambda^{-(2+\frac{1}{k})^{-1}},$$

$$(1.6) \quad \|T_\lambda^h\| \leq \text{const } \lambda^{-\frac{n-1}{2}} h^{\frac{1}{2k}+\frac{1}{2K}}, \quad \lambda^{-\frac{1}{2}} \leq h \leq \lambda^{-(2+\frac{1}{K})^{-1}},$$

where the constants depend only on the bounds on derivatives of ψ and S in (1.2) (up to some finite order).

Note that our methods are only applicable in the region $h \geq \lambda^{-\frac{1}{2}}$ (in a certain sense, this is the restriction due to the uncertainty principle).

Remark. The estimate (1.6) is also applicable to T_λ localized to the region where $|h(x, \vartheta)| \leq 2h$ (that is, when we no longer require $|h(x, \vartheta)| \geq h/2$). We will denote such an operator by \bar{T}_λ^h :

$$(1.7) \quad \bar{T}_\lambda^h u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \bar{\beta}(h^{-1}h(x, \vartheta)) u(\vartheta) d\vartheta, \quad \bar{\beta} \in C_{\text{comp}}^\infty([-2, 2]).$$

Let the functions β and $\bar{\beta}$ satisfy $\sum_{\pm} \sum_{j=1}^\infty \beta(\pm 2^{-j}t) + \bar{\beta}(t) = 1$, for any $t \in \mathbb{R}$. Then we can decompose T_λ as

$$(1.8) \quad T_\lambda = \sum_{\pm} \sum_{h > h_0} T_\lambda^{\pm h} + \bar{T}_\lambda^{h_0}, \quad h = 2^{-N}, \quad N \in \mathbb{N},$$

where the cut-off value h_0 is to be chosen properly. We can apply Theorem 1.1 to each T_λ^h (the estimates (1.3)-(1.5)) and to $\bar{T}_\lambda^{h_0}$ (the estimate (1.6)). This gives the following estimate on T_λ :

Corollary 1. *Under the assumptions of Theorem 1.1,*

$$(1.9) \quad \|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + \sup_p \delta},$$

where “the loss in the rate of decay at a point p ” is given by

$$\delta(l, r) = \frac{1}{2} \left(1 - \frac{1}{2 \min(l, r)} \right) \left(1 + \frac{1}{2 \max(l, r)} \right)^{-1},$$

with l and r being the types of π_L and π_R at p . The supremum in (1.9) is taken over all points of \mathcal{C} . There is certainly no loss of smoothness at non-singular points: we define $\delta(0, 0) = 0$.

This is weaker (except when $l = 1$ or $r = 1$) than the optimal result (proved in [PhSt97] for $n = 1$) which we might expect: $\delta_{\text{opt}}(l, r) = \frac{1}{2} \left(1 + \frac{1}{l} + \frac{1}{r} \right)^{-1}$.

Damping estimates. We can use Theorem 1.1 for deriving the damping estimates. According to the estimates (1.3)–(1.6), the series $\sum_{\pm} \sum_{\hbar > \lambda^{-\frac{1}{2}}} \hbar \|T_{\lambda}^{\pm \hbar}\| + \lambda^{-\frac{1}{2}} \|\bar{T}_{\lambda}^{\lambda^{-\frac{1}{2}}}\|$ (where \hbar varies dyadically, as in (1.8)), is bounded by $\text{const } \lambda^{-\frac{n}{2}}$. Hence, if U_{λ} is an operator like (1.1) but with a damping factor of magnitude $\leq \text{const } |\det S_{x\vartheta}|$, then $\|U_{\lambda}\| \leq \text{const } \lambda^{-n/2}$. This proves the following result:

Corollary 2. *Let U_{λ} be a compactly supported oscillatory integral operator of the form (1.1) with singularities of finite type on both sides. If the density ψ vanishes on the critical variety $\Sigma = \{(x, \vartheta) \mid \det S_{x\vartheta}(x, \vartheta) = 0\}$ so that $|\psi| \leq \text{const } |\det S_{x\vartheta}|$, then U_{λ} has the same decay of its $L^p \rightarrow L^p$ norm as non-singular oscillatory integral operators:*

$$(1.10) \quad \|U_{\lambda}\|_{L^p \rightarrow L^p} \leq \text{const } \lambda^{-\frac{n}{2} + n|\frac{1}{p} - \frac{1}{2}|}, \quad 1 \leq p \leq \infty.$$

Note that we have interpolated the L^2 estimates with the trivial L^1 and L^{∞} estimates (which are uniform in λ).

According to [GrSe⁹⁴], the L^2 estimate in Corollary 2 implies the analogous result for Fourier integral operators:

Corollary 3. *Let $A \in I^m(X, Y, \mathcal{C})$ be a Fourier integral operator associated to a canonical relation \mathcal{C} such that the projections π_L, π_R are of corank at most 1 and have finite types everywhere. If the symbol of A vanishes on the critical variety of π_L and π_R , $|\sigma(A)| \leq \text{const } |\det d\pi|$, $d\pi$ being the Jacobi matrix of either of π_L, π_R , then for any real s*

$$A : H_{\text{comp}}^s(Y) \rightarrow H_{\text{loc}}^{s-m}(X).$$

L^p estimates. Let us say a few words about $L^p \rightarrow L^p$ estimates on oscillatory integral operators. They can be derived by interpolating L^2 estimates with $L^1 \rightarrow L^1$ and $L^{\infty} \rightarrow L^{\infty}$ estimates:

Theorem 1.2. *Let $\text{rank } S_{x\vartheta} \geq n - 1$. If T_{λ} has singularities of type at most l on the left and at most r on the right, then*

$$(1.11) \quad \|T_{\lambda}^{\pm \hbar}\|_{L^1 \rightarrow L^1} \leq \text{const } \hbar^{\frac{1}{r}},$$

$$(1.12) \quad \|T_{\lambda}^{\pm \hbar}\|_{L^{\infty} \rightarrow L^{\infty}} \leq \text{const } \hbar^{\frac{1}{l}}.$$

The same estimates are satisfied for $\bar{T}_{\lambda}^{\hbar}$.

We may apply Theorems 1.1, 1.2 to derive the $L^p \rightarrow L^p$ estimates on T_{λ} . Both L^2 and L^1, L^{∞} estimates on $\bar{T}_{\lambda}^{\hbar}$ become better for smaller values of \hbar ; by interpolation, we see that this is also true for L^p estimates for any $1 \leq p \leq \infty$. The estimates

on T_λ^\hbar have a more complicated behavior: L^1 , L^∞ estimates become better for smaller values of \hbar , while L^2 estimates “blow up” as $\hbar \rightarrow 0$. Therefore, $L^p \rightarrow L^p$ estimates on T_λ^\hbar improve as $\hbar \rightarrow 0$ only if p is outside a certain neighborhood of $p = 2$. In this case, the norms on the operators in the dyadic decomposition (1.8) only become better as \hbar becomes smaller, and we conclude that the estimate on the entire T_λ is determined by operators which are truncated off the critical variety (large values of \hbar) and hence coincides with the norm of non-degenerate oscillatory integral operators.

In a certain neighborhood of $p = 2$, we need to glue the diverging L^p estimates on T_λ^\hbar with the estimate on $\bar{T}_\lambda^{\hbar_0}$, at some point $\hbar_0 > \lambda^{-\frac{1}{2}}$.

At some “boundary values” of p , the L^p estimates on T_λ^\hbar are neither improving nor blowing up when \hbar becomes small. Therefore, all the terms in (1.8) have the same bounds, and we are getting a factor $\ln \hbar_0^{-1} \sim \ln \lambda$ (this is the number of terms in (1.8)).

The L^p estimates we obtain in this fashion are optimal only if the canonical relation associated to T_λ has a Whitney fold at least on one side (this is when we know the optimal $L^2 \rightarrow L^2$ estimates [Co⁹⁸]):

Corollary 4. *Let T_λ be a compactly supported oscillatory integral operator with a fold singularity on the left. If the singularity on the right is of type at most r , then the operator T_λ has the same continuity properties in L^p , for $p < \frac{r+2}{r+1}$ and for $p > 3$, as a non-singular oscillatory integral operator:*

$$(1.13) \quad \|T_\lambda\|_{L^p \rightarrow L^p} \leq \text{const } \lambda^{-\frac{n}{2} + n|\frac{1}{p} - \frac{1}{2}|}, \quad 1 \leq p < \frac{r+2}{r+1}, \quad 3 < p \leq \infty.$$

For $\frac{r+2}{r+1} \leq p \leq 3$ the estimates are obtained by the interpolation with the L^2 estimates,

$$(1.14) \quad \|T_\lambda\|_{L^p \rightarrow L^p} \leq \text{const } \lambda^{-\frac{n}{2} + (4 + \frac{2}{r})^{-1}}.$$

These estimates are sharp for $\frac{r+2}{r+1} < p < 3$. At the endpoints $p = \frac{r+2}{r+1}$ and $p = 3$, we can only prove weak estimates (with the extra factor $\ln \lambda$).

Remark. The estimates in Theorem 1.2 may be improved if certain additional conditions on the projections are satisfied, and this in turn leads to the estimate (1.13) in Corollary 4 to be true for a wider range of values of p . For example, if $d_x(\det S_{x\vartheta})|_\Sigma \neq 0$ (this condition is satisfied if π_R has a strong $S_{1_r}^+$ -singularity, in the sense of [GrSe^{97a}]), then $\|T_\lambda^\hbar\|_{L^1 \rightarrow L^1} \leq \text{const } \hbar$, and then one can easily prove that the estimate (1.13) is valid in the range $1 \leq p < \frac{3}{2}$ and $3 < p \leq \infty$. We will not discuss this issue here.

L^p - L^q estimates. There are two more estimates which hold for both $T_\lambda^{\pm\hbar}$ and \bar{T}_λ^{\hbar} :

$$\|T_\lambda^{\pm\hbar}\|_{L^1 \rightarrow L^\infty} \leq \text{const},$$

which is trivially satisfied, and

$$\|T_\lambda^{\pm\hbar}\|_{L^\infty \rightarrow L^1} \leq \text{const } \hbar,$$

which is satisfied if $d_{x,\vartheta}(\det S_{x\vartheta})|_\Sigma \neq 0$. The interpolation yields a variety of L^p - L^q estimates on T_λ which we do not discuss. For the case of oscillatory integral operators with two-sided Whitney folds, see [GrSe^{97b}].

We will prove Theorem 1.2 in Section 2 and Theorem 1.1 in Sections 3, 4, and 5.

2. USING FINITE TYPE CONDITIONS: L^1 AND L^∞ ESTIMATES

We illustrate how the finite type conditions work on the simplest example: we derive L^1 and L^∞ estimates formulated in Theorem 1.2. We need certain preparation: We choose local coordinates $x = (x', x_n)$ and $\vartheta = (\vartheta', \vartheta_n)$ so that $S_{x'\vartheta'}$ is non-degenerate (recall that the corank of $S_{x\vartheta}$ is at most 1), and consider the map $\pi_R|_\vartheta$ as a composition

$$(2.1) \quad (x', x_n) \xrightarrow{\pi'} (\eta' = S_{\vartheta'}(x, \vartheta), x_n) \xrightarrow{\pi^s} (\eta', \eta_n = S_{\vartheta_n}).$$

According to the condition of the theorem that π_R is of type at most r , we may assume that

$$(2.2) \quad \mathbf{K}_R^r h \geq \varkappa_R > 0,$$

where the vector field $\mathbf{K}_R = (\partial_{x_n})_{\eta' = S_{\vartheta'} \text{ fixed}}$ has the property $\mathbf{K}_R|_{h(x,\vartheta)=0} \in \text{Ker } d\pi_R$. Its explicit form is

$$(2.3) \quad \mathbf{K}_R = (\partial_{x_n})_{\eta' = S_{\vartheta'} \text{ fixed}} = \partial_{x_n} - S^{\vartheta'x'} S_{x_n\vartheta'} \partial_{x'},$$

where $S^{\vartheta'x'}(x, \vartheta)$ is the inverse to the matrix $S_{x'\vartheta'}$ at a point (x, ϑ) .

Proof of Theorem 1.2. We will write the generic notation β for the localizing functions $\beta, \bar{\beta}$; the argument is the same for both $T_\lambda^{\pm\hbar}$ and \bar{T}_λ^{\hbar} . The key property of these operators is the small size of the support of their integral kernels “in the critical direction”; we are to estimate this size using the finite type conditions.

We have:

$$\begin{aligned} \|T_\lambda^{\hbar} u\|_{L^1} &\leq \iint dx d\vartheta |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta)) u(\vartheta)| \\ &\leq \|u\|_{L^1} \cdot \text{const} \sup_{\vartheta} \int dx |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta))|. \end{aligned}$$

The bound which we need for the proof of (1.11),

$$(2.4) \quad \int dx |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta))| \leq \text{const } \hbar^{\frac{1}{r}},$$

is due to the assumptions that the map $(x, \vartheta) \mapsto (\vartheta, S_\vartheta)$ is of type r . We change the variables of integration to $\eta' = S_{\vartheta'}(x, \vartheta)$ and $t = x_n$:

$$\int dx |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta))| = \int \frac{d\eta'}{|\det S_{x' \vartheta'}|} dt |\psi \beta(\hbar^{-1} h)|.$$

We claim that the integration with respect to t contributes $\text{const } \hbar^{\frac{1}{r}}$ (while the integration with respect to η' is over the compact domain). Indeed, since $\partial_t = (\partial_{x_n})_{\eta'} = \mathbf{K}_R$, we know from (2.2) that $\partial_t^r h \geq \varkappa_R > 0$. Now everything follows from the following lemma:

Lemma 2.1. *Let $h \in C^r(\mathbb{R})$ be a function such that $|h^{(r)}(t)| \geq \varkappa > 0$ for t in some interval $I \subset \mathbb{R}$. Then the set $I^\hbar = \{t \in I \mid |h(t)| < \hbar\}$ consists of at most 2^{r-1} intervals I_σ^\hbar , possibly with joint ends, with each of them being of measure $|I_\sigma^\hbar| \leq (2r!/\varkappa)^{\frac{1}{r}} \hbar^{\frac{1}{r}}$.*

This lemma is well-known; see, e.g., [Ch⁸⁵]. Let us give a proof which also motivates the partition of 1 which will follow in Section 3.

Proof. First, we take σ to be a set of $r-1$ signs, $\sigma = (\sigma_1, \dots, \sigma_{r-1})$, $\sigma_j = \pm 1$. We define

$$I_\sigma = \{t \in I \mid \sigma_j h^{(j)}(t) \geq 0, j = 1, \dots, r-1\}.$$

Clearly, $I = \cup_\sigma I_\sigma$. Since $h^{(r)}$ does not change sign, $h^{(r-1)}$ is monotone and hence the set $\{t \in I \mid \sigma_{r-1} h^{(r-1)} \geq 0\}$ is connected. Continuing by induction, we conclude that I_σ is also connected. We now define $I_\sigma^\hbar = I^\hbar \cap I_\sigma$, which is also connected (since h is monotone on each I_σ). We parameterize I_σ^\hbar by t , so that t changes from 0 to $\delta \equiv |I_\sigma^\hbar|$. Then for $0 \leq t \leq \delta$ we have:

$$\text{either } \sigma_{r-1} h^{(r-1)}(t) \geq \varkappa t \quad \text{or} \quad \sigma_{r-1} h^{(r-1)}(t) \geq \varkappa \delta - t.$$

The rest is by induction; we will arrive at

$$\text{either } \sigma_1 h'(t) \geq \frac{\varkappa}{(r-1)!} t^{r-1} \quad \text{or} \quad \sigma_1 h'(t) \geq \frac{\varkappa}{(r-1)!} (\delta - t)^{r-1},$$

and hence $|h(\delta) - h(0)| \geq \frac{\varkappa}{r!} \delta^r$. The *a priori* bound $|h(\delta) - h(0)| < 2\hbar$ gives the desired estimate on $\delta = |I_\sigma^\hbar|$. \square

For the $L^\infty \rightarrow L^\infty$ estimate, we derive

$$\begin{aligned} \|T_\lambda^\hbar u\|_{L^\infty} &\leq \sup_x \int d\vartheta |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta)) u(\vartheta)| \\ &\leq \|u\|_{L^\infty} \cdot \text{const} \sup_x \int d\vartheta |\psi(x, \vartheta) \beta(\hbar^{-1} h(x, \vartheta))|. \end{aligned}$$

As above, we may prove that if the map $(x, \vartheta) \mapsto (x, S_x)$ is of type l , then $\int d\vartheta |\psi \beta(\hbar^{-1} h)| \leq \text{const} \hbar^{\frac{1}{l}}$, and the estimate (1.12) follows. This completes the proof of Theorem 1.2 \square

3. LOCALIZATIONS

We are now going to prove Theorem 1.1. Thus, let both π_L and π_R be of corank at most 1 and have finite types. We assume that on the support of the integral kernel of T_λ the types of the projections are at most l and r , respectively. Our statements for the cases $l = 1$ or $r = 1$ already follow from [Co97], so we assume that $l > 1$ and $r > 1$. For the definiteness, we will also assume that $l \geq r$.

The argument is the same for $T_\lambda^{\pm\hbar}$; for our convenience we will always consider T_λ^\hbar (that is, we always assume that $\det S_{x\vartheta}$ is positive).

We split the integral kernel of T_λ^\hbar into pieces, in the spirit of the proof of Lemma 2.1. For this, we pick a smooth function ρ , $\text{supp } \rho \subset [-1, \infty)$, such that $\rho(t) + \rho(-t) = 1$, and introduce the following partition of 1:

$$\begin{aligned} 1 &= \sum_{\sigma} \rho_{\sigma}^{\hbar}(x, \vartheta), \quad \sigma = (\sigma_1, \dots, \sigma_{l-1}), \quad \sigma_j = \pm 1, \\ (3.1) \quad \rho_{\sigma}^{\hbar}(x, \vartheta) &= \prod_{j=1}^{l-1} \rho(\hbar^{-1} \sigma_j \mathbf{K}_R^j h(x, \vartheta)). \end{aligned}$$

Here the vector field $\mathbf{K}_R = \partial_{x_n} - S^{\vartheta'x'} S_{x_n \vartheta'} \partial_{x'}$ is the same as in (2.3).

Analogously, we introduce the partition

$$\begin{aligned} 1 &= \sum_{\varsigma} \varrho_{\varsigma}^{\hbar}(x, \vartheta), \quad \varsigma = (\varsigma_1, \dots, \varsigma_{r-1}), \quad \varsigma_j = \pm 1, \\ (3.2) \quad \varrho_{\varsigma}^{\hbar}(x, \vartheta) &= \prod_{j=1}^{r-1} \rho(\hbar^{-1} \varsigma_j \mathbf{K}_L^j h(x, \vartheta)), \quad \mathbf{K}_L = \partial_{\vartheta_n} - S^{\vartheta'x'} S_{x' \vartheta_n} \partial_{\vartheta'}. \end{aligned}$$

Of course, $\mathbf{K}_L|_{h(x, \vartheta)=0} \in \text{Ker } d\pi_L$.

Note that (i) the summation indexes σ and ς in (3.1), (3.2) take finitely many values, and (ii) these are *admissible* partitions, in the sense that

$$|\partial_x^\alpha \partial_\vartheta^\beta \rho_{\sigma}^{\hbar}(x, \vartheta)| \leq C_{\alpha\beta} \hbar^{-|\alpha| - |\beta|},$$

so that only \hbar^{-1} can be contributed during integrations by parts which will follow later in the argument. We continue the proof individually for each of the pieces of T_λ^\hbar with fixed σ, ς .

We use the “fine” partitions of 1,

$$(3.3) \quad \begin{aligned} 1 &= \sum_{X \in \mathbb{Z}^n} \chi(\hbar^{-1}\eta' - X') \chi(\hbar^{-1}x_n - X_n), & \eta' &\equiv S_{\vartheta'}(x, \vartheta), \\ 1 &= \sum_{\Theta \in \mathbb{Z}^n} \chi(\hbar^{-1}\xi' - \Theta') \chi(\hbar^{-1}\vartheta_n - \Theta_n), & \xi' &\equiv S_{x'}(x, \vartheta), \end{aligned}$$

where χ is a certain smooth function supported in the unit ball in \mathbb{R}^n . Multiplying the integral kernel of the operator T_λ^\hbar by the above functions, we decompose T_λ^\hbar into $T_\lambda^\hbar = \sum_{X, \Theta \in \mathbb{Z}^n} (T_\lambda^\hbar)_{X\Theta}$.

Convexity. We will use the fact that the map $\pi_R|_\vartheta : x \mapsto S_\vartheta(x, \vartheta)$ (and similarly $\pi_L|_x$) satisfies certain convexity condition: Given ϑ , then for any x, y on a connected set where $\det S_{x\vartheta} \geq \hbar/2$ the following inequality holds:

$$(3.4) \quad |S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)| \geq \text{const } \hbar |x - y|.$$

Let us show (sketching the argument from [Co⁹⁷]) that the property (3.4) is satisfied on the support of each of σ, ς -pieces of T_λ^\hbar . The map π' in (2.1) is a diffeomorphism and hence we may assume that $|\eta'(x) - \eta'(y)| \geq \text{const } |\det S_{x'\vartheta'}| \cdot |x - y|$. We now need to investigate the map $\pi^s|_{\eta'} : x_n \mapsto \eta_n = S_{\vartheta_n}(x, \vartheta)$. Let us denote by \mathcal{L} the line segment from (η', x_n) to (η', y_n) . We have:

$$(3.5) \quad |\eta_n(\eta', y_n) - \eta_n(\eta', x_n)| \geq |y_n - x_n| \cdot \inf_{\mathcal{L}} |(\partial_{x_n})_{\eta'} \eta_n|$$

We need to show that the factor at $|y_n - x_n|$ in the right-hand side of (3.5) is of magnitude \hbar , and then the inequality (3.4) follows.

The value of the derivative $(\partial_{x_n})_{\eta'} \eta_n$ can be determined from the decomposition $\pi_R|_\vartheta = \pi_R^s|_\vartheta \circ \pi_R'|_\vartheta$. Considering the determinants of the Jacobi matrices in (2.1), $J(\pi_R^s|_\vartheta) \cdot J(\pi_R'|_\vartheta) = J(\pi_R|_\vartheta)$, we obtain $(\partial_{x_n})_{\eta'} \eta_n \cdot \det S_{x'\vartheta'} = h(x, \vartheta)$. Hence,

Lemma 3.1. *There is the relation $(\partial_{x_n})_{\eta'} \eta_n = \frac{h(x, \vartheta)}{\det S_{x'\vartheta'}}$.*

We also need to check that $h(x, \vartheta) \geq \text{const } \hbar$ on a line between x_n and y_n :

Lemma 3.2. *If $|\mathcal{L}| \leq \text{const}$, then $h \geq \frac{\hbar}{4}$ everywhere on \mathcal{L} .*

Hence, we admit that the line segment \mathcal{L} could be not entirely on the support of the integral kernel of T^\hbar , where $h \geq \hbar/2$.

Proof. Let t be a parameter on the line segment \mathcal{L} , changing from $t = 0$ at the point (η', x_n) to $t = |x_n - y_n|$ at the point (η', y_n) . We consider $h(x, \vartheta)|_{\mathcal{L}}$ as a function of

t ; $\partial_t = (\partial_{x_n})_{\eta'}$. Since both (η', x_n) and (η', y_n) are on the support of σ , ς -piece of T_λ^h , we know that at $t = 0$ and at $t = |x_n - y_n|$

$$(3.6) \quad h \geq \hbar/2,$$

$$(3.7) \quad \sigma_j \cdot (\partial_t)^j h \geq -\hbar, \quad \forall j \in \mathbb{N}, \quad j < r.$$

Due to the finite type conditions on both projections from \mathcal{C} , we also know that $(\partial_t)^r h > 0$ (or instead < 0), for all t between 0 and $|x_n - y_n|$ (see (2.2)).

If we assumed that in (3.6) and (3.7) $\hbar = 0$, then we would conclude by induction that all $(\partial_t)^j h(t)$, $j < r$, were monotone functions which did not change the signs between $t = 0$ and $t = |x_n - y_n|$, and hence $h(t)$ would be concluded monotone (see the proof of Lemma 2.1). Since $\hbar \neq 0$, the above conclusion is true modulo the error of magnitude \hbar ; hence, the function $h(t)$ is “almost monotone” (its derivative is greater than $-\text{const } \hbar$ or less than $\text{const } \hbar$), therefore the value of $h(t)$ can not drop below $\hbar/4$ as long as t is between 0 and $|x_n - y_n|$ and as long as at the boundary points the value of h is not less than $\hbar/2$. We also need to assume that $|x_n - y_n|$ is not too large. \square

4. ALMOST ORTHOGONALITY RELATIONS FOR DIFFERENT PIECES

We are going to apply the Cotlar-Stein lemma on L^2 almost orthogonality [St⁹³]. For our convenience, let us formulate this result here.

Cotlar-Stein Lemma. *Let E and F be the Hilbert spaces, and let $\{T_i \mid i \in \mathbb{Z}\}$ be a family of continuous operators $E \rightarrow F$ which satisfy the following conditions:*

$$(4.1) \quad \|T_i^* T_j\| \leq a(i, j), \quad \|T_i T_j^*\| \leq b(i, j),$$

where $a(i, j)$ and $b(i, j)$ are non-negative functions on $\mathbb{Z} \times \mathbb{Z}$. If a and b satisfy

$$(4.2) \quad A \equiv \sup_i \sum_j a^{\frac{1}{2}}(i, j) < \infty, \quad B \equiv \sup_i \sum_j b^{\frac{1}{2}}(i, j) < \infty,$$

then the formal sum $\sum_i T_i$ converges (in the weak operator topology) to a continuous operator $T : E \rightarrow F$, which is bounded by

$$(4.3) \quad \|T\| \leq A^{\frac{1}{2}} B^{\frac{1}{2}}.$$

The details of the proof are in [St⁹³].

Now we are going to investigate the almost orthogonality relations for the operators $(T_\lambda^h)_{X\Theta}$, with respect to different multi-indexes $\{X\Theta\}$.

Almost orthogonality with respect to different Θ , W . Let us consider the behavior of the compositions $(T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)^*_{YW}$ with respect to different Θ and W . We will show that if X and Y are fixed and if W is also fixed, then the composition $(T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)^*_{YW}$ is different from zero only for finitely many values of Θ . The integral kernel of such an operator is given by

$$(4.4) \quad K_{\Theta W}(x, y) = \int_{\mathbb{R}^n} d\vartheta \chi(\hbar^{-1}S_{x'}(x, \vartheta) - \Theta') \chi(\hbar^{-1}S_{y'}(y, \vartheta) - W') \\ \chi(\hbar^{-1}S_{\vartheta'}(x, \vartheta) - X') \chi(\hbar^{-1}S_{w'}(y, \vartheta) - Y') \times \dots,$$

and in addition we know that $x_n \approx \hbar X_n$, $y_n \approx \hbar Y_n$, and $\vartheta_n \approx \hbar \Theta_n \approx \hbar W_n$. (Each time, the error is at most \hbar .) Let us consider the following system:

$$(4.5) \quad \begin{cases} S_{y'}(y', y_n, \vartheta', \vartheta_n) = \hbar W', \\ S_{\vartheta'}(y', y_n, \vartheta', \vartheta_n) = \hbar Y'. \end{cases}$$

Given y_n, ϑ_n, Y' , and W' , we can solve this system for y' and ϑ' , since the matrix $\frac{\partial(\hbar W', \hbar Y')}{\partial(y', \vartheta')} = \begin{bmatrix} S_{y'\vartheta'} & S_{\vartheta'\vartheta'} \\ S_{y'y'} & S_{\vartheta'y'} \end{bmatrix}$ is non-degenerate. (For this, we could have used certain preparation: at some point (x_o, ϑ_o) on the support of T_λ we make the matrices $S_{x'x'}$ and $S_{\vartheta'\vartheta'}$ vanish, using the change $S(x, \vartheta) \mapsto S(x, \vartheta) + \Phi_1(x') + \Phi_2(\vartheta')$, which is equivalent to a unitary transformation. Possibly, we also use a restriction to a smaller neighborhood of (x_o, ϑ_o) .) Now, since the parameters y_n and ϑ_n ($\approx \hbar Y_n$ and $\approx \hbar \Theta_n$) and also the right-hand sides of the system (4.5) are determined with the error \hbar , both y' and ϑ' are determined with the error of the same magnitude. Then, from $S_{\vartheta'}(x', x_n, \vartheta) \approx \hbar X'$, $x_n \approx \hbar X_n$, we determine x' (same error). Hence, $\hbar \Theta' \approx S_{x'}(x, \vartheta)$ is also determined with the error of magnitude \hbar . We conclude that Θ can take only finitely many values (uniformly in λ, \hbar). Note that the particular range of Θ may depend on the specific values of X, Y , and W .

Almost orthogonality with respect to different X, Y . The almost orthogonality of compositions $(T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)^*_{YW}$ with respect to different X and Y requires the integration by parts in the expression for the integral kernel:

$$(4.6) \quad K \left((T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)^*_{YW} \right) (x, y) = \int_{\mathbb{R}^n} e^{i\lambda(S(x, \vartheta) - S(y, \vartheta))} \chi(\dots) \beta(\dots) \psi(\dots) d^n \vartheta.$$

Integration by parts in the expression (4.6) shows that for any integer N

$$(4.7) \quad \left| K \left((T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)^*_{YW} \right) (x, y) \right| \leq \text{const}_N \int \frac{d^n \vartheta \chi(\dots) \beta(\dots)}{[1 + \lambda \hbar |S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^{2N}}.$$

The factor \hbar in the denominator reflects the contribution of \hbar^{-1} from each integration by parts (to estimate the contribution of certain terms, one needs to refer to (3.4)).

We claim that

$$|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)| \geq \text{const}(\hbar|X' - Y'| + \hbar^2|X_n - Y_n|).$$

For $\hbar|X_n - Y_n| \leq |X' - Y'|$ this inequality is trivial, while for $\hbar|X_n - Y_n| \geq |X' - Y'|$ we use the convexity property (3.4) of π_R :

$$|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)| \geq \text{const} \hbar|x - y| \geq \text{const} \hbar^2|X_n - Y_n|.$$

Now we may rewrite the right-hand side of (4.7) as

$$\frac{\text{const}}{[1 + \lambda\hbar(\hbar|X' - Y'| + \hbar^2|X_n - Y_n|)]^N} \int \frac{d^n \vartheta \chi(\dots) \beta(\dots)}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N}$$

and apply the Schur lemma:

$$\begin{aligned} \left\| (T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)_{YW}^* \right\| &\leq \int d^n x \left| K \left((T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)_{YW}^* \right) (x, y) \right| \\ &\leq \frac{\text{const}}{[1 + \lambda\hbar(\hbar|X' - Y'| + \hbar^2|X_n - Y_n|)]^N} \int \frac{d^n x d^n \vartheta \chi(\dots) \beta(\dots)}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N}. \end{aligned}$$

We will integrate in x first. If $\hbar \geq \lambda^{-\frac{1}{3}}$, then

$$\int \frac{d^n x}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N} = \int \frac{d^n \{S_\vartheta\}}{|\det S_{x\vartheta}|} \cdot \frac{1}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N}$$

is bounded by $\text{const}(\lambda\hbar)^{-n} \hbar^{-1}$. If instead $\hbar \leq \lambda^{-\frac{1}{3}}$, then a better bound is obtained when appealing the size of the support in x_n :

$$\int \frac{d^n x \chi(\dots) \beta(\dots)}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N} = \int \frac{d^{n-1} \{S_{\vartheta'}\}}{|\det S_{x'\vartheta'}|} \cdot \frac{dx_n \chi(\hbar^{-1}x_n - X_n) \dots}{[1 + \lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|]^N}.$$

This expression is bounded by $\leq \text{const}(\lambda\hbar)^{-n+1} \hbar$: $(\lambda\hbar)^{-n+1}$ is due to the integration in $S_{\vartheta'}$, and \hbar is due to the integration in x_n .

The integration with respect to ϑ is performed as follows:

$$\int d\vartheta^n \chi(\dots) = \int \frac{d^{n-1} \{S_{y'}\} d\vartheta_n}{|\det S_{y'\vartheta'}|} \chi(\hbar^{-1}S_{y'}(y, \vartheta) - \Theta') \chi(\hbar^{-1}\vartheta_n - \Theta_n) \leq \text{const} \hbar^n.$$

We conclude that

$$(4.8) \quad \| (T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)_{YW}^* \| \leq \text{const } \tau^2 [1 + \lambda \hbar^2 (|X' - Y'| + \hbar |X_n - Y_n|)]^{-N},$$

where $\tau^2 = \min(\lambda^{-n} \hbar^{-1}, \lambda^{-n+1} \hbar^2)$. One may think of τ as of the L^2 -estimate on a generic operator $(T_\lambda^\hbar)_{X\Theta}$.

Let us rewrite (4.8) as

$$(4.9) \quad \| (T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)_{YW}^* \| \leq \tau^2 a(X, \Theta; Y, W),$$

where $a(X, \Theta; Y, W) \in C(\mathbb{Z}^{4n}; \mathbb{R}_+)$ is some function which “measures” the orthogonality of operators. Similarly to (4.2), we define

$$(4.10) \quad A = \sup_{Y, W} \sum_{X, \Theta} a^{\frac{1}{2}}(X, \Theta; Y, W);$$

roughly, this is “the number of the operators which are not orthogonal” (imagine that a takes values 0 and 1 only). We proved earlier that for fixed X , Y , and W , the multi-index Θ takes only finitely many values; therefore the summation with respect to Θ in (4.10) is over a finite region in \mathbb{Z}^n and only contributes some factor which is uniform in \hbar and λ .

Before we proceed to the analysis of the summation in X , let us say a few words about the compositions of the form T^*T . For such compositions, we have estimates similar to (4.9):

$$(4.11) \quad \| (T_\lambda^\hbar)_{X\Theta}^* (T_\lambda^\hbar)_{YW} \| \leq C \tau^2 b(X, \Theta; Y, W);$$

we define

$$(4.12) \quad B = \sup_{Y, W} \sum_{X, \Theta} b^{\frac{1}{2}}(X, \Theta; Y, W).$$

By the symmetry, we know that the summation with respect to X in (4.12) is over a finite subset in \mathbb{Z}^n .

We analyze the summation in X in four different cases: $\hbar \geq \lambda^{-\frac{1}{3}}$, $\lambda^{-(2+\frac{1}{r})^{-1}} \leq \hbar \leq \lambda^{-\frac{1}{3}}$, $\lambda^{-(2+\frac{1}{l})^{-1}} \leq \hbar \leq \lambda^{-(2+\frac{1}{r})^{-1}}$, and $\lambda^{-\frac{1}{2}} \leq \hbar \leq \lambda^{-(2+\frac{1}{r})^{-1}}$.

The case $\hbar \geq \lambda^{-\frac{1}{3}}$. The simplest case is when $\hbar \geq \lambda^{-\frac{1}{3}}$; then (4.8) decreases faster than τ^2 times any power of $|X - Y|^{-1}$, and the sum \sum_X in (4.10) is bounded uniformly in \hbar , λ , so that $A \leq \text{const}$. By the symmetry, B in (4.12) is also uniformly bounded.

According to the Cotlar-Stein lemma (4.3), the bound on T_λ^\hbar is given by

$$\tau = \min(\lambda^{-\frac{n-1}{2}} \hbar, \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}),$$

which is the root of the common factor in (4.9), (4.11), times the geometric mean of A and B :

$$(4.13) \quad \|T_\lambda^\hbar\| \leq \tau \sqrt{AB} \leq \text{const } \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}} \sqrt{AB} \leq \text{const } \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}, \quad \hbar \geq \lambda^{-\frac{1}{3}}.$$

The case $\lambda^{-(2+\frac{1}{r})^{-1}} \leq \hbar \leq \lambda^{-\frac{1}{3}}$. Again, for each X , the summation in Θ in $A = \sum_{X,\Theta} a^{\frac{1}{2}}(X, \Theta; Y, W)$ is over a finite set of multi-indices. But, if $\lambda\hbar^3 \leq 1$, then it follows from (4.8) that the summation with respect to X in (4.10) contributes

$$\sum_X [1 + \lambda\hbar^2|X' - Y'| + \lambda\hbar^3|X_n - Y_n|]^{-N} \leq \text{const}(\lambda\hbar^3)^{-1}.$$

Note that the summation with respect to X' in (4.10) is fine (contributes a factor uniform in λ, \hbar) as long as $\lambda\hbar^2 \geq 1$. Our conclusion is that “the number of non-orthogonal operators” is controlled by

$$(4.14) \quad A = \sum_{X,\Theta} a^{\frac{1}{2}}(X, \Theta; Y, W) \leq \text{const}(\lambda\hbar^3)^{-1}.$$

Similarly, $B = \sum_{X,\Theta} b^{\frac{1}{2}}(X, \Theta; Y, W) \leq \text{const}(\lambda\hbar^3)^{-1}$, and (4.13) becomes

$$(4.15) \quad \|T_\lambda^\hbar\| \leq \text{const} \lambda^{-\frac{n-1}{2}} \hbar \sqrt{AB} = \text{const} \lambda^{-\frac{n+1}{2}} \hbar^{-2}, \quad \lambda^{-\frac{1}{2}} \leq \hbar \leq \lambda^{-\frac{1}{3}}.$$

The case $\lambda^{-(2+\frac{1}{t})^{-1}} \leq \hbar \leq \lambda^{-(2+\frac{1}{r})^{-1}}$. The estimate (4.15) is clearly inadequate for small values of \hbar : its derivation is based on the assumption that $\det S_{x\vartheta} \neq 0$, and as a consequence the estimate blows up when $\hbar \rightarrow 0$ and does not allow to estimate the contribution of some tiny neighborhood of the critical variety $\{\det S_{x\vartheta} = 0\}$. Let us get another estimate on A , trying to count “non-orthogonal terms” directly. In (4.14), we evaluated the sum assuming that the number of terms with different X_n is infinite, while certainly X_n takes at most $\text{const} \hbar^{-1}$ values. More than that, if the projection π_R is of type r , then there are only $\text{const} \hbar^{-1+\frac{1}{r}}$ terms with different X_n . This is because the size of support of the integral kernel of T_λ^\hbar is not only compact, but also bounded in certain critical direction by $\hbar^{\frac{1}{r}}$ (this is gained by the methods which are very much the same as in Section 2; we will show this in more detail in Section 5). This leads to the following bound on A in (4.10):

$$(4.16) \quad A \leq \text{const} \hbar^{-1+\frac{1}{r}}.$$

Now we proceed to deriving the resulting bounds on T_λ^\hbar . If $\hbar \leq \lambda^{-(2+\frac{1}{r})^{-1}}$, then (4.16) gives a better bound on A than (4.14). We assume that $\hbar \geq \lambda^{-(2+\frac{1}{t})^{-1}}$, so that a proper bound on B is still $(\lambda\hbar^3)^{-1}$. Hence, we can rewrite (4.13) as

$$(4.17) \quad \|T_\lambda^\hbar\| \leq \text{const} \lambda^{-\frac{n-1}{2}} \hbar \sqrt{\hbar^{-1+\frac{1}{r}} (\lambda\hbar^3)^{-1}} \leq \text{const} \lambda^{-\frac{n}{2}} \hbar^{-1+\frac{1}{2r}}.$$

The case $\lambda^{-\frac{1}{2}} \leq \hbar \leq \lambda^{-(2+\frac{1}{t})^{-1}}$. In this region, the best bounds on both A and B are due to the finiteness of types of the projections π_L and π_R : $A \leq \text{const} \hbar^{-1+\frac{1}{r}}$, $B \leq \text{const} \hbar^{-1+\frac{1}{t}}$. This gives

$$(4.18) \quad \|T_\lambda^\hbar\| \leq \text{const} \lambda^{-\frac{n-1}{2}} \hbar \sqrt{\hbar^{-1+\frac{1}{r}} \hbar^{-1+\frac{1}{t}}} \leq \text{const} \lambda^{-\frac{n-1}{2}} \hbar^{\frac{1}{2t}+\frac{1}{2r}}.$$

Since the derivation of bounds $A \leq \text{const} \hbar^{-1+\frac{1}{r}}$ and $B \leq \text{const} \hbar^{-1+\frac{1}{t}}$ does not appeal to the inequality $\det S_{x\vartheta} \geq \hbar/2$ (for details, see Section 5), we conclude that \bar{T}_λ^\hbar also satisfies the estimate (4.18).

5. ALMOST ORTHOGONALITY AND FINITE TYPE CONDITIONS

We are left to prove the bound (4.16). As in Section 2, we consider the map $\pi_R|_{\vartheta}$ as decomposed into

$$(5.1) \quad (x', x_n) \xrightarrow{\pi'} (\eta' = S_{\vartheta'}(x, \vartheta), x_n) \xrightarrow{\pi^s} (\eta', \eta_n = S_{\vartheta_n}).$$

Since π_R is of type at most r , we may assume that on the support of the integral kernel of the operator T_λ we have a uniform bound

$$(5.2) \quad (\partial_{x_n})_{\eta'}^r h \geq \varkappa_R > 0.$$

We work in the space (η', x_n) . Consider the line segment \mathcal{L} which connects the points $(\eta'(x, \vartheta), x_n)$ and $(\eta'(y, \vartheta), y_n)$. There are two cases:

- The line segment \mathcal{L} is *outside* the $c\hbar^{1-\frac{1}{r}}$ -cone of the direction $\pm(\partial_{x_n})_{\eta'}$ (c should be sufficiently small; see later); this corresponds to

$$(5.3) \quad |S_{\vartheta'}(x, \vartheta) - S_{\vartheta'}(y, \vartheta)| \geq c\hbar^{1-\frac{1}{r}}|x_n - y_n|.$$

We derive that the gradient of the phase function, $\lambda(S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta))$, is bounded in the absolute value from below by

$$\lambda|S_{\vartheta'}(x, \vartheta) - S_{\vartheta'}(y, \vartheta)| \geq \frac{1}{2}\lambda \left(|S_{\vartheta'}(x, \vartheta) - S_{\vartheta'}(y, \vartheta)| + c\hbar^{1-\frac{1}{r}}|x_n - y_n| \right).$$

Similarly to how we arrived at (4.8), we derive that for any integer N

$$\| (T_\lambda^\hbar)_{X\Theta} (T_\lambda^\hbar)_{YW}^* \| \leq \text{const } \lambda^{-n+1} \hbar^2 \left[1 + \lambda \hbar^2 \left(|X' - Y'| + c\hbar^{1-\frac{1}{r}}|X_n - Y_n| \right) \right]^{-N}.$$

Since $\lambda \hbar^2 \geq 1$, the summation \sum_{X_n} in (4.10) contributes at most

$$\sum_{X_n} \left[1 + c\lambda \hbar^{3-\frac{1}{r}}|X_n - Y_n| \right]^{-N} \leq \text{const } c^{-1} \hbar^{-1+\frac{1}{r}},$$

in an agreement with (4.16). (Recall at this point that the summation with respect to Θ in (4.10) is over a bounded set in \mathbb{Z}^n and that the summation with respect to X' converges, as long as $\hbar \geq \lambda^{-\frac{1}{2}}$.)

- Now assume that the line segment \mathcal{L} from $(\eta'(x), x_n)$ to $(\eta'(y), y_n)$ is *inside* the $c\hbar^{1-\frac{1}{r}}$ -cone of the directions $\pm(\partial_{x_n})_{\eta'}$. If $|x - y| \geq \text{const } \hbar^{\frac{1}{r}}$, then, using (5.2), we will show that

$$(5.4) \quad |h(x, \vartheta) - h(y, \vartheta)| \sim \varkappa_R(\hbar|\mathcal{L}|)^r,$$

where $|\mathcal{L}| \equiv \text{dist}[(\eta'(x), x_n), (\eta'(y), y_n)]$. Since the left-hand side of (5.4) can not be greater than $4\hbar$ (the value of $|h|$ at both points (x, ϑ) and (y, ϑ) is bounded by $2\hbar$), we gain the bound $|\mathcal{L}| \leq \text{const } \hbar^{\frac{1}{r}}$. This, together with $|\mathcal{L}| \geq |x_n - y_n| \approx \hbar |X_n - Y_n|$, yields the desired restriction

$$(5.5) \quad |X_n - Y_n| \leq \text{const } \hbar^{-1+\frac{1}{r}},$$

which again leads to (4.16).

The detailed proof of (5.5) is in [Co⁹⁸]; for the reader's convenience, we give here the sketch. Let t be a parameter on the line segment \mathcal{L} , which changes from $t = 0$ at $\pi'(x)$ to $t = |\mathcal{L}|$ at $\pi'(y)$. We consider h as a function of t . Since \mathcal{L} is in the $c\hbar^{1-\frac{1}{r}}$ -cone of $\pm(\partial_{x_n})_{\eta'}$,

$$\partial_t^j h - \mathbf{K}_R^j h = O(c\hbar^{1-\frac{1}{r}}).$$

At the points (x, ϑ) and (y, ϑ) , $\sigma_j \mathbf{K}_R^j h \geq -\hbar$; we assume c is so small that

$$(5.6) \quad \sigma_j h^{(j)}(t) \geq -\hbar^{1-\frac{1}{r}} \quad \text{at the points } t = 0 \text{ and } t = |\mathcal{L}|.$$

Since $|\mathbf{K}_R^r h| \geq \varkappa_R$, we also know that (again, assuming that c is sufficiently small)

$$(5.7) \quad |h^{(r)}(t)| \geq \frac{\varkappa_R}{2} > 0, \quad \text{for any } t \text{ between } 0 \text{ and } |\mathcal{L}|.$$

If we assumed that $\hbar = 0$ in the right-hand side of (5.6), then, similarly to the argument in the proof of Lemma 2.1, we would conclude that the derivatives $h^{(j)}(t)$ of all orders $j < r$ were monotone functions which did not change the signs between $t = 0$ and $|\mathcal{L}|$. Moreover, we would derive that $|h(|\mathcal{L}|) - h(0)| \geq \frac{\varkappa_R}{2} \cdot \frac{|\mathcal{L}|^r}{r!}$. Since at the endpoints of \mathcal{L} the values of $\sigma_j h^{(j)}$ are only greater than $-\hbar^{1-\frac{1}{r}}$, there is an error involved; its magnitude is bounded by $O(\hbar^{1-\frac{1}{r}}|\mathcal{L}|)$. We arrive at

$$(5.8) \quad |h(|\mathcal{L}|) - h(0)| \geq \frac{\varkappa_R}{2} \cdot \frac{|\mathcal{L}|^r}{r!} - \text{const } \hbar^{1-\frac{1}{r}}|\mathcal{L}|.$$

Therefore, $|\mathcal{L}| \leq \text{const } \hbar^{\frac{1}{r}}$, and this proves (5.5).

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